REAL OPTION VALUE, CH 12
MULTI-FACTOR SWITCHING OPTIONS

Switching options are often embedded in facilities, or situations, sometimes developed with the imagination and initiative of the participant, or manager. For instance, Miss Lucy Steele obtained the best of (her perceived) two outcomes in marrying Robert Ferrars, after he had received the family fortune, abandoning her previous engagement to Edward Ferrars[1].

This chapter presents three basic two factor multiple switching option models: (i) switching to the highest price output; (ii) switching from an operating state with an option to suspend operations, or from a suspended state to an operating state, when both output price and input cost are stochastic; and (iii) switching to the lowest cost input. As a simplification (and reduction in the number of equations required for a solution), a single switch, such as from operating to suspended, is also considered. The model for the best of two outputs is adapted from Dockendorf and Paxson (2011), for suspension and restart options from Adkins and Paxson (2012), and for two inputs from Adkins and Paxson (2011).

1 GENERAL SWITCHING OPTIONS

When is the right time for an operator of a flexible facility to switch back and forth between two possible outputs or inputs in order to maximise value when switching costs are taken into account? Which factors should be monitored in making these decisions? How much should an investor pay for such a flexible operating asset? What are the strategy implications for the operator, investor and possibly for policy makers?

Flexible production and processing facilities are typically more expensive to operate, and with a higher initial investment cost, than inflexible facilities. One problem is that one part of the flexible facility, which requires an additional

[1] Jane Austen, Sense and Sensibility, 1811. Edward Ferrars's inheritance is settled on his younger brother Robert, when his mother hears of Lucy's engagement to Edward. Lucy promptly transfers her affection, following the money, apparently a single switch.

investment cost, might be idle at times. Investing in a facility which is not productive all the time seems counter-intuitive at first glance. What is frequently misunderstood is that the additional option value through “operating flexibility” (Trigeorgis and Mason, 1987) may have significant value in uncertain markets and when there is less than perfect correlation between input factors, or between possible outputs, or indeed between inputs and outputs. Examples of flexible assets include shipping (combination carriers), the chemical industry (flexible fertilizer plants), electricity generation (system switching from coal to natural gas), and real estate (multiple property uses).

The traditional approach to determine switching boundaries between two operating modes is to discount future cash flows and use Jevons-Marshallian present value triggers. This methodology does not fully capture the option value which may arise due to the uncertainty in future input or output prices. The value of waiting to gain more information on future price or cost developments, and consequently on the optimal switching triggers can be best viewed in a real options framework.

Conceptually, the switch between two volatile assets or commodities can be modelled as an exchange option. Margrabe (1978) and McDonald and Siegel (1986) model European and American perpetual exchange options, respectively, which are linear homogeneous in the underlying stochastic variables. An analytical model for flexible production capacity is presented by He and Pindyck (1992), where switching costs and product-specific operating costs are ignored, thereby eliminating the components which would lead to a non-linearity of the value function in the underlying processes. Brekke and Schieldrop (2000) also assume costless switching in their study on the value of operating flexibility between two stochastic input factors, in which they determine the optimal investment timing for a flexible technology in comparison to a technology that does not allow input switching. Adkins and Paxson (2011) present quasi-analytical solutions to input switching options, where two-factor functions are not homogeneous of degree one, and thus dimension reducing techniques used in McDonald and Siegel (1986) and Paxson and Pinto (2005) are not available. Other approaches, such as Song et al. (2010), consider the net profits (revenue less cost) or spreads in the respective operating state to be stochastic, reducing two factors to one.

Brennan and Schwartz (1985) consider switching states from idle to operating, operating to suspension, and then back, based on one stochastic factor. Paxson (2005) extends the solution for up to eight different state options, each with a distinct trigger, but for only one stochastic factor.

Geltner, Riddiough and Stojanovic (1996) develop a framework for a perpetual option on the best of two underlying assets, applied to the case of two alternative uses for properties, and provide a comprehensive discussion of relevant assumptions for such a contingent-claims problem. Childs, Riddiough and Triantis (1996) extend this model to allow for redevelopment or switching between alternative uses.

The next section presents two real option models for an asset with switching opportunities between two outputs with uncertain prices, taking into account switching costs and operating costs. The first model is a quasi-analytical solution for multiple switching among the best of two outputs; the second for single one-way switching.

2 Multiple Output Switching

2.1 Assumptions

Consider a flexible facility which can be used to produce one of two different outputs by switching between operating modes. Assume the prices of the two outputs, x and y, are stochastic and correlated and follow geometric Brownian motion (gBm):

$$dx = (\mu_x - \delta_x) x\, dt + \sigma_x\, x\, dz_x \quad (1)$$

$$dy = (\mu_y - \delta_y) y\, dt + \sigma_y\, y\, dz_y \quad (2)$$

with the notations:

μ Required return on the output

δ Convenience yield of the output

σ Volatility of the output

dz Wiener process (stochastic element)

ρ Correlation between the two output prices: $dz_x\, dz_y / dt$

The instantaneous cash flow in each operating mode is the respective commodity price of the output product less unit operating cost, assuming production of one (equivalent) unit per annum, $(x - c_x)$ in operating mode '1' and $(y - c_y)$ in operating mode '2'. The operating costs c_x and c_y are per unit produced. A switching cost of S_{12} is incurred when switching from operating mode '1' to '2', and S_{21} for switching back. The appropriate discount rate is r for non- stochastic elements, such as constant operating costs. For convenience and simplicity, assume that the appropriate discount rate for stochastic variables is δ which is equal to μ-r.

Further assumptions are that the operating costs are deterministic and constant, the lifetime of the asset is infinite, and the company is not restricted in the product mix choice because of selling commitments. Moreover, the typical assumptions of real options theory apply, with interest rates, convenience yields, volatilities and correlation constant over time.

2.2 Quasi-analytical Solution for Continuous Switching

The asset value with opportunities to continuously switch between the two operating modes is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let V_1 be the asset value in operating mode '1', producing output x, and V_2 the asset value in operating mode '2', producing output y accordingly. The switching options depend on the two correlated stochastic variables x and y, and so do the asset value functions which are defined by the following partial differential equations:

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_1}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_1}{\partial x} + (r-\delta_Y)y\frac{\partial V_1}{\partial y} - rV_1 + (x - c_X) = 0 \tag{3}$$

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_2}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_2}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_2}{\partial x} + (r-\delta_Y)y\frac{\partial V_2}{\partial y} - rV_2 + (y - c_y) = 0 \tag{4}$$

Two-factor problems which are linear homogeneous, i.e. $\mathrm{V}(\lambda \cdot \mathrm{x}; \lambda \cdot \mathrm{y}) = \lambda \cdot \mathrm{V}(\mathrm{x}; \mathrm{y})$, can typically be solved analytically by substitution of variables, so that the partial differential equation can be reduced to a one-factor differential equation. An example of this is the perpetual American exchange option in McDonald and Siegel (1986). With constant switching cost, operating cost and multiple switching, the problem is no longer homogenous of degree one and the dimension reducing technique cannot be used.

Dockendorf and Paxson (2011) derive a quasi-analytical solution for a similar type of two-factor non-homogeneous problem. For two outputs, the partial differential equations are satisfied by the following general solutions:

$$V_1(x,y) = \frac{x}{\delta_x} - \frac{c_x}{r} + A x^{\beta_{11}} y^{\beta_{12}} \tag{5}$$

where β_{11} and β_{12} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1) + \tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1) + \rho\sigma_x\sigma_y\beta_{11}\beta_{12} + \beta_{11}(r-\delta_x) + \beta_{12}(r-\delta_y) - r = 0, \tag{6}$$

and

$$V_2(x,y) = \frac{y}{\delta_y} - \frac{c_y}{r} + B x^{\beta_{21}} y^{\beta_{22}} \tag{7}$$

where β_{21} and β_{22} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{21}(\beta_{21}-1) + \tfrac{1}{2}\sigma_y^2\beta_{22}(\beta_{22}-1) + \rho\sigma_x\sigma_y\beta_{21}\beta_{22} + \beta_{21}(r-\delta_x) + \beta_{22}(r-\delta_y) - r = 0 \tag{8}$$

The characteristic root equation (6) is solved by combinations of β_{11} and β_{12} forming an ellipse of such form that β_{11} could be positive or negative and β_{12} could be positive or negative. The same is true for equation (8). Since the option to switch from x to y decreases with x and increases with y, β_{11} must be negative and β_{12} positive. Likewise, β_{21} must be positive and β_{22} negative. Switching between the operating modes always depends on the level of both x and y. At the switching points (x_{12}, y_{12}) and (x_{21}, y_{21}), the asset value in the current operating mode must be equal to the asset value in the alternative operating mode net of switching cost. These value matching conditions are stated formally below:

$$A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}} + \frac{x_{12}}{\delta_x} - \frac{c_x}{r} = B x_{12}^{\beta_{21}} y_{12}^{\beta_{22}} + \frac{y_{12}}{\delta_y} - \frac{c_y}{r} - S_{12} \tag{9}$$

$$A x_{21}^{\beta_{11}} y_{21}^{\beta_{12}} + \frac{x_{21}}{\delta_x} - \frac{c_x}{r} - S_{21} = B x_{21}^{\beta_{21}} y_{21}^{\beta_{22}} + \frac{y_{21}}{\delta_y} - \frac{c_y}{r} \tag{10}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11} A x_{12}^{\beta_{11}-1} y_{12}^{\beta_{12}} + \frac{1}{\delta_x} = \beta_{21} B x_{12}^{\beta_{21}-1} y_{12}^{\beta_{22}} \tag{11}$$

$$\beta_{12} A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}-1} = \beta_{22} B x_{12}^{\beta_{21}} y_{12}^{\beta_{22}-1} + \frac{1}{\delta_y} \tag{12}$$

$$\beta_{11} A x_{21}^{\beta_{11}-1} y_{21}^{\beta_{12}} + \frac{1}{\delta_x} = \beta_{21} B x_{21}^{\beta_{21}-1} y_{21}^{\beta_{22}} \tag{13}$$

$$\beta_{12} A x_{21}^{\beta_{11}} y_{21}^{\beta_{12}-1} = \beta_{22} B x_{21}^{\beta_{21}} y_{21}^{\beta_{22}-1} + \frac{1}{\delta_y} \tag{14}$$

There are only 8 equations, (6) and (8) - (14), for 10 unknowns, β_{11}, β_{12}, β_{21}, β_{22}, A, B, x_{12}, y_{12}, x_{21}, y_{21}, so there is no completely analytical solution. Yet, for every value of x, there has to be a corresponding value of y when switching should occur, (x_{12}, y_{12}) from x to y and (x_{21}, y_{21}) from y to x. So a quasi-analytical solution can be found by assuming values for x, which then solves the set of simultaneous equations for all remaining variables, given that $x = x_{12} = x_{21}$. This procedure is repeated for many values of x, providing the corresponding option values and the switching boundaries.

The spread between the two switching boundaries can be viewed in term of the wedges, defined below.

$$\frac{y_{12}}{\delta_y} \cdot \Omega_{y_{12}} - \frac{x_{12}}{\delta_x} \cdot \Omega_{x_{12}} = S_{12} \tag{15}$$

$$\frac{x_{21}}{\delta_x} \cdot \Omega_{x_{21}} - \frac{y_{21}}{\delta_y} \cdot \Omega_{y_{21}} = S_{21} \tag{16}$$

The Marshallian rule is satisfied when all wedges (Ω_{x12}, Ω_{x21}, Ω_{y12}, Ω_{y21}) are equal to one. The wedges for the real option model are:

$$\Omega_{x_{12}} = \Omega_{x_{21}} = 1 - \frac{\beta_{12} - \beta_{22}}{\beta_{12}\beta_{21} - \beta_{11}\beta_{22}} \tag{17}$$

$$\Omega_{y_{12}} = \Omega_{y_{21}} = 1 - \frac{\beta_{21} - \beta_{11}}{\beta_{12}\beta_{21} - \beta_{11}\beta_{22}} \tag{18}$$

Since β_{12} and β_{21} are positive and β_{11} and β_{22} are negative and the denominator of (17) and (18) needs to be positive to justify the option values, the wedges are less than one. This demonstrates that the switching hysteresis (band of inaction, no switching) is larger than suggested by the Marshallian rule.

It can be seen by rearranging (9) and (10) that the net difference in the switching options including the value of outputs at the triggers is the sum of the switching cost and the difference in the present value of operating costs, $S_{12}+\left(\frac{c_y-c_x}{r}\right)$ and $S_{21}-\left(\frac{c_y-c_x}{r}\right)$, respectively.

2.3 Quasi-analytical Solution for One-Way Switching

The solution for the asset value with a one-way switching option from the above model with continuous switching is straight-forward. Assuming $c_y \geq c_x$, the American perpetual option to switch from x to y can be determined. The asset value V_{1S} is given by (5) with the characteristic root equation (22), and V_{2S} is given by (7) with B=0, thereby eliminating the option to switch back. Applying the same solution procedure as before, a quasi-analytical solution is obtained.

$$A\,x_{12}^{\beta_{11}}\,y_{12}^{\beta_{12}}+\frac{x_{12}}{\delta_x}-\frac{c_x}{r}=\frac{y_{12}}{\delta_y}-\frac{c_y}{r}-S_{12} \tag{19}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11}A\,x_{12}^{\beta_{11}-1}\,y_{12}^{\beta_{12}}+\frac{1}{\delta_x}=0 \tag{20}$$

$$\beta_{12}\,A\,x_{12}^{\beta_{11}}\,y_{12}^{\beta_{12}-1}-\frac{1}{\delta_y}=0 \tag{21}$$

where β_{11} and β_{12} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1)+\tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1)+\rho\sigma_x\sigma_y\beta_{11}\beta_{12}+\beta_{11}(r-\delta_x)+\beta_{12}(r-\delta_y)-r=0\,, \tag{22}$$

The characteristic root equation (22) together with value matching condition (19) and smooth pasting conditions (20) and (21) represents the system of 4 equations, while there are 5 unknowns, β_{11}, β_{12}, A, x_{12}, y_{12}.

Numerical Illustrations

Here are illustrative results for the multiple and single output switch models, assuming current operating costs are half of current gross revenue for each output. Figure 1 shows that the option factors A and B are positive, β_{11} and β_{22} are negative and β_{12} and β_{21} are positive, thereby fulfilling the requirements from the theoretical model. The system of value matching conditions, smooth pasting conditions and characteristic root equations is fully satisfied.

Figure 1

	A	B	C	D
1	**Multiple American Perpetual Output Switch Option**			
2				
3	OUTPUT X	x	100	
4	OUTPUT Y	y	100	
5	Convenience yield of x	δ_x	0.04	
6	Convenience yield of y	δ_y	0.04	
7	Volatility of x	σ_x	0.40	
8	Volatility of y	σ_y	0.30	
9	Correlation x with y	ρ	0.50	
10	Risk-free interest rate	r	0.05	
11	Operating cost for x	c_x	50	
12	Operating cost for y	c_y	50	
13				
14				
15	Switching cost from x to y	S_{12}	50	
16	Switching cost from y to x	S_{21}	70	
17	PV of revenues x	X	1,500	LHS EQ5
18	PV of revenues y	Y	1,500	LHS EQ7
19	Switching boundary X to Y	x_{12}	100	
20	Switching boundary Y to X	x_{21}	100	
21		SOLUTION		OPTION VALUE
22	Asset value in operating mode '1'	**$V_1(x,y)$**	**2,780.37**	**1,280**
23	Asset value in operating mode '2'	**$V_2(x,y)$**	**2,781.76**	**1,282**
24		A	10.17	
25		B	9.76	
26	Switching boundary x to y	y_{12} (x)	**150.21**	
27	Switching boundary x to y	y_{21} (x)	**53.18**	
28	Solution quadrant	β_{11}	-0.4012	*must be negative*
29	Solution quadrant	β_{12}	1.4512	*must be positive*
30	Solution quadrant	β_{21}	1.4229	*must be positive*
31	Solution quadrant	β_{22}	-0.3636	*must be negative*
32		EQUATIONS		
33	Value matching 1	EQ9	0.000	
34	Value matching 2	EQ10	0.000	
35	Smooth pasting 1A	EQ11	0.000	
36	Smooth pasting 1B	EQ12	0.000	
37	Smooth pasting 2A	EQ13	0.000	
38	Smooth pasting 2B	EQ14	0.000	
39	Solution quadrant 1	EQ6	0.000	
40	Solution quadrant 2	EQ8	0.000	
41		Sum	0.000	
42		SOLVER: SET C40=0, CHANGING C24:C31		
43		Δ	1.9190	C29*C30-C28*C31
44		Wedge x	0.0543	1-(C29-C31)/C43
45		Wedge y	0.0495	1-(C30-C28)/C43
46		EQ 15	0.0000	(C26/C6)*C45-(C19/C5)*C44-C15
47		EQ 16	0.0000	(C20/C5)*C44-(C27/C6)*C45-C16
48	EQ9	(C24*C19^C28*C26^C29+C19/C5-C11/C10-C25*C19^C30*C26^C31-C26/C6+C12/C10+C15)		
49	EQ10	(C24*C20^C28*C27^C29+C20/C5-C11/C10-C25*C20^C30*C27^C31-C27/C6+C12/C10-C16)		
50	EQ11	(C28*C24*C19^(C28-1)*C26^C29+1/C5-C30*C25*C19^(C30-1)*C26^C31)		
51	EQ12	(C29*C24*C19^C28*C26^(C29-1)-C31*C25*C19^C30*C26^(C31-1)-1/C6)		
52	EQ13	(C28*C24*C20^(C28-1)*C27^C29+1/C5-C30*C25*C20^(C30-1)*C27^C31)		
53	EQ14	(C29*C24*C20^C28*C27^(C29-1)-C31*C25*C20^C30*C27^(C31-1)-1/C6)		
54	EQ6	0.5*C7^2*C28*(C28-1)+0.5*C8^2*C29*(C29-1)+C9*C7*C8*C28*C29+C28*(C10-C5)+C29*(C10-C6)-C10		
55	EQ8	0.5*C7^2*C30*(C30-1)+0.5*C8^2*C31*(C31-1)+C9*C7*C8*C30*C31+C30*(C10-C5)+C31*(C10-C6)-C10		
56	SPREAD		97.04	
57	$V_1(x,y)$ EQ 5	C3/C5-C11/C10+C24*C3^C28*C4^C29		
58	$V_2(x,y)$ EQ 7	C4/C6-C12/C10+C25*C3^C30*C4^C31		

The asset values are given in both operating modes, V_1 and V_2, and the level of y is indicated when it is optimal to switch from x to y (y_{12}) and vice versa (y_{21}). With x and y having the same initial values and the same convenience yields, the asset value with no switching is identical in both operating modes when the operating cost is the same. Higher operating costs reduce the asset value. When operating costs are 50, the asset value V_1 with continuous switching opportunities is valued at 2780 if the incumbent is x_{12}=100 with a volatility of 40% according to the quasi-analytical solution. The switching option value is the difference between the asset value and the value with no switching option, 2780-1500=1280, and 2782-1500=1282 for V_2. The option to switch between the two operating modes adds about 85% to the inflexible asset value. Switching to output y is justified if y increases to 50% higher than output x, and back to x, if the y output price falls to half of the x price. The spread between y_{12} and y_{21} is due to switching costs and stochastic elements, and increases with high volatilities and low correlation, following real options theory. It should be noted that changing x also changes the switching boundaries y_{12} and y_{21}, and that the switching boundaries x_{12} and x_{21} for a given level of y can be determined in a similar way. The fact that y_{12} and y_{21} are not symmetrical to x = 100 is primarily due to the log-normality of the prices, and further due to $S_{12} \neq S_{21}$ and $\sigma_x \neq \sigma_y$.

Figure 2

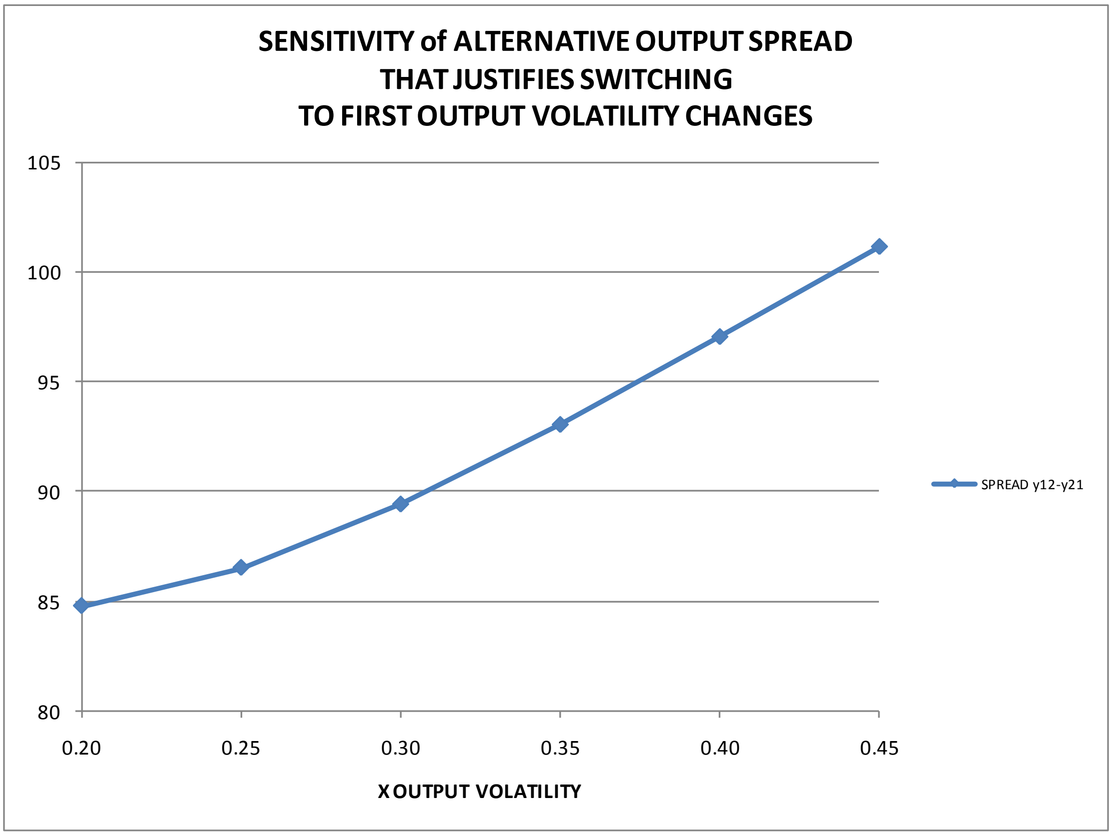


Figure 2 illustrates the sensitivity of the switching boundaries of the quasi-analytical solution for continuous switching to changes in x output price volatility. Switching boundaries are further apart when volatilities are higher. This is consistent with general real option theory because uncertainty is taken into account which delays switching in order to gain more information. In contrast to this, the Marshallian rule stipulates that switching is optimal as soon as the present value of expected cash flows after switching exceeds the present value of expected cash flows before switching by the switching cost.

Figure 3

	A	B	C	D	E	F	G	H
1	**Continuous American Perpetual SINGLE SWITCH Option**							
2	ONE WAY SWITCH FROM OUTPUT X TO Y							
3	OUTPUT X	x	100					
4	OUTPUT Y	y	100					
5	Convenience yield of x	δ_x	0.04					
6	Convenience yield of y	δ_y	0.04					
7	Volatility of x	σ_x	0.40					
8	Volatility of y	σ_y	0.30					
9	Correlation x with y	ρ	0.50					
10	Risk-free interest rate	r	0.05					
11	Operating cost for x	c_x	50					
12	Operating cost for y	c_y	50					
13	Switching cost from x to y	S_{12}	50					
14								
15	PV of revenues x	X	1,500	LHS EQ5				
16	PV of revenues y	Y	1,500	LHS EQ 7				
17	Switching boundary x to y	x_{12}	100					
18								
19		SOLUTION		OPTION VALUE				
20	Asset value in operating mode '1'	**V_1(x,y)**	**2,528.98**	**1,029**				
21	Asset value in operating mode '2'	**V_2(x,y)**	**1,500.00**	0				
22		A	9.89					
23	Switching boundary x to y	y_{12} (x)	**337.04**					
24	Solution quadrant	β_{11}	-0.4255	*must be negative*				
25	Solution quadrant	β_{12}	1.4340	*must be positive*				
26		EQUATIONS						
27	Value matching 1	EQ 19	0.000					
28	Smooth pasting 1A	EQ 20	0.000					
29	Smooth pasting 1B	EQ 21	0.000					
30	Solution quadrant 1	EQ 22	0.000					
31		Sum	0.000					
32		SOLVER: SET C31=0, CHANGING C22:C25						
33	EQ 19	(C22*C17^C24*C23^C25+C17/C5-C11/C10-C23/C6+C12/C10+C13)						
34	EQ 20	(C24*C22*C17^(C24-1)*C23^C25+1/C5)						
35	EQ 21	(C25*C22*C17^C24*C23^(C25-1)-1/C6)						
36	EQ 22	0.5*C7^2*C24*(C24-1)+0.5*C8^2*C25*(C25-1)+C9*C7*C8*C24*C25+C24*(C10-C5)+C25*(C10-C6)-C10						
37	SPREAD		**237.04**					
38	V_1(x,y) EQ 5	C3/C5-C11/C10+C22*C3^C24*C4^C25						
39	V_2(x,y) EQ 7	C4/C6-C12/C10						

When switching is only possible from x to y but not vice versa, the switching trigger y_{12S} is much (224%) higher as shown in Figure 3 because the decision cannot be reversed. The asset value V_{1S} is 9% lower compared to multiple switching.

3 Quasi-analytical Solution for Continuous Input-Output Switching

The asset value with opportunities to continuously switch between an operating mode and a suspended mode (when both inputs and outputs are stochastic) is given by the present value of perpetual cash flows in the current operating mode plus the option to switch to the alternative mode. Let V_1 be the asset value in operating mode '1', producing output x at input cost y, and V_2 the asset value in a suspension mode '2'. The switching options depend on the two correlated stochastic variables x and y, and so do the asset value functions which are defined by the following partial differential equations:

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_1}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_1}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_1}{\partial x} + (r-\delta_Y)y\frac{\partial V_1}{\partial y} - rV_1 + (x-y) = 0 \tag{23}$$

$$\frac{1}{2}\sigma_X^2 x^2 \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{2}\sigma_Y^2 y^2 \frac{\partial^2 V_2}{\partial y^2} + \rho\sigma_X\sigma_Y xy \frac{\partial^2 V_2}{\partial x \partial y} + (r-\delta_X)x\frac{\partial V_2}{\partial x} + (r-\delta_Y)y\frac{\partial V_2}{\partial y} - rV_2 = 0 \tag{24}$$

The operating mode has the production income (x-y) and the option to suspend; the suspension mode has only the option to re-start operations. For stochastic outputs and inputs, the partial differential equations are satisfied by the following general solutions:

$$V_1(x,y) = \frac{x}{\delta_x} - \frac{y}{\delta_y} + Ax^{\beta_{11}}y^{\beta_{12}} \tag{25}$$

where β_{11} and β_{12} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1) + \tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1) + \rho\sigma_x\sigma_y\beta_{11}\beta_{12} + \beta_{11}(r-\delta_x) + \beta_{12}(r-\delta_y) - r = 0\,, \tag{26}$$

and

$$V_2(x,y) = Bx^{\beta_{21}}y^{\beta_{22}} \tag{27}$$

where β_{21} and β_{22} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{21}(\beta_{21}-1) + \tfrac{1}{2}\sigma_y^2\beta_{22}(\beta_{22}-1) + \rho\sigma_x\sigma_y\beta_{21}\beta_{22} + \beta_{21}(r-\delta_x) + \beta_{22}(r-\delta_y) - r = 0 \tag{28}$$

Since the option to switch from operating to suspension decreases with x and increases with y, β_{11} must be negative and β_{12} positive. Likewise, β_{21} must be positive and β_{22} negative. Switching between the operating and suspension modes always depends on the level of both x and y. At the switching points (x_{12}, y_{12}) and (x_{21}, y_{21}), the asset value in the current operating mode must be equal to the asset value in the alternative operating mode net of switching cost. These value matching conditions are:

$$V_1\left(x_{12}, y_{12}\right) = V_2(x_{12}, y_{12}) - S_{12}$$

$$V_2\left(x_{21}, y_{21}\right) = V_1(x_{21}, y_{21}) - S_{21}$$

$$A\, x_{12}{}^{\beta_{11}}\, y_{12}{}^{\beta_{12}} + \frac{x_{12}}{\delta_x} - \frac{y_{12}}{\delta_y} = B\, x_{12}{}^{\beta_{21}}\, y_{12}{}^{\beta_{22}} - S_{12} \tag{29}$$

$$B\, x_{21}{}^{\beta_{21}}\, y_{21}{}^{\beta_{22}} = A\, x_{21}{}^{\beta_{11}}\, y_{21}{}^{\beta_{12}} + \frac{x_{21}}{\delta_x} - \frac{y_{21}}{\delta_y} - S_{21} \tag{30}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11} A x_{12}{}^{\beta_{11}-1}\, y_{12}{}^{\beta_{12}} + \frac{1}{\delta_x} = \beta_{21} B\, x_{12}{}^{\beta_{21}-1}\, y_{12}{}^{\beta_{22}} \tag{31}$$

$$\beta_{12}\, A\, x_{12}{}^{\beta_{11}}\, y_{12}{}^{\beta_{12}-1} - \frac{1}{\delta_y} = \beta_{22}\, B\, x_{12}{}^{\beta_{21}}\, y_{12}{}^{\beta_{22}-1} \tag{32}$$

$$\beta_{21}\, B\, x_{21}{}^{\beta_{21}-1}\, y_{21} = \beta_{11}\, A\, x_{21}{}^{\beta_{11}-1}\, y_{21}{}^{\beta_{12}} + \frac{1}{\delta_x} \tag{33}$$

$$\beta_{22}\, B x_{21}{}^{\beta_{21}}\, y_{21}{}^{\beta_{22}-1} = \beta_{12}\, A x_{21}{}^{\beta_{11}}\, y_{21}{}^{\beta_{12}-1} - \frac{1}{\delta_y} \tag{34}$$

There are only 8 equations, (26) and (28- 34), for 10 unknowns, β_{11}, β_{12}, β_{21}, β_{22}, A, B, x_{12}, y_{12}, x_{21}, y_{21}, so there is no completely analytical solution. Yet, for every value of x, there has to be a corresponding value of y when switching should occur, (x_{12}, y_{12}) and (x_{21}, y_{21}). So a quasi-analytical solution can be found by assuming values for x, which then solves the set of simultaneous equations for all remaining variables, given that $x = x_{12} = x_{21}$. This procedure is repeated for many values of x, providing the corresponding option values and the switching boundaries.

Single Switch

The solution for the asset value with a one-way switching option from the above model with multiple switching is straight-forward. This one-way switch constitutes an abandonment option, where the switching cost is the abandonment cost. The asset value V_{1S} is given by (25) with the characteristic root equation (38), and V_{2S} is given by (27) with B=0, thereby eliminating the option to switch back. Applying the same solution procedure as before, a quasi-analytical solution is obtained.

$$A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}} + \frac{x_{12}}{\delta_x} - \frac{y_{12}}{\delta_y} + S_{12} = 0 \tag{35}$$

Furthermore, smooth pasting conditions hold at the boundaries:

$$\beta_{11} A x_{12}^{\beta_{11}-1} y_{12}^{\beta_{12}} + \frac{1}{\delta_x} = 0 \tag{36}$$

$$\beta_{12} A x_{12}^{\beta_{11}} y_{12}^{\beta_{12}-1} - \frac{1}{\delta_y} = 0 \tag{37}$$

where β_{11} and β_{12} satisfy the characteristic root equation

$$\tfrac{1}{2}\sigma_x^2\beta_{11}(\beta_{11}-1)+\tfrac{1}{2}\sigma_y^2\beta_{12}(\beta_{12}-1)+\rho\sigma_x\sigma_y\beta_{11}\beta_{12}+\beta_{11}(r-\delta_x)+\beta_{12}(r-\delta_y)-r=0\,, \tag{38}$$

The characteristic root equation (38) together with value matching condition (35) and smooth pasting conditions (36) and (37) represents the system of 4 equations, while there are 5 unknowns, β_{11}, β_{12}, A, x_{12}, y_{12}.

Figure 4

	A	B	C	D	E	F	G	H	I	J
1	**Continuous American Perpetual IN-OUT Exchange Option Non-homogeneous**									
2		INPUT			IN_OUT_JAN2011					
3	PRICE	x	100							
4	COST	y	50							
5	Convenience yield of x	δ_x	0.04							
6	Convenience yield of y	δ_y	0.04							
7	Volatility of x	σ_x	0.40							
8	Volatility of y	σ_y	0.30							
9	Correlation x with y	ρ	0.50							
10	Risk-free interest rate	r	0.05							
11										
12										
13										
14										
15	Switching cost from x to y	S_{12}	50							
16	Switching cost from y to x	S_{21}	70							
17										
18										
19	Switching boundary OP to SHUT	x_{12}	100							
20	Switching boundary SHUT to OP	x_{21}	100							
21		SOLUTION		OPTION	OPERATING					
22	Asset value in operating mode	**V_1(x,y)**	**1718.27**	**468.27**	**1250.00**					
23	Asset value in shut down mode	**V_2(x,y)**	**1649.16**	**1649.16**						
24		A	10.17							
25		B	9.76							
26	Switching boundary OP to SHUT	y_{12} (x)	**150.21**							
27	Switching boundary SHUT to OP	y_{21} (x)	**53.18**							
28	Solution quadrant	β_{11}	-0.40116	*must be negative*						
29	Solution quadrant	β_{12}	1.45116	*must be positive*						
30	Solution quadrant	β_{21}	1.42289	*must be positive*						
31	Solution quadrant	β_{22}	-0.36360	*must be negative*						
32		EQUATIONS								
33	Value matching 1	EQ 29	0.000	(C24*C19^C28*C26^C29+C19/C5-C26/C6-C25*C19^C30*C26^C31+C15)						
34	Value matching 2	EQ 30	0.000	(C25*C20^C30*C27^C31-C24*C20^C28*C27^C29-C20/C5+C27/C6+C16)						
35	Smooth pasting 1A	EQ 31	0.000	(C28*C24*(C19^(C28-1))*C26^C29+1/C5-C30*C25*(C19^(C30-1))*C26^C31)						
36	Smooth pasting 1B	EQ 32	0.000	(C29*C24*C19^C28*(C26^(C29-1))-1/C6-C31*C25*C19^C30*(C26^(C31-1)))						
37	Smooth pasting 2A	EQ 33	0.000	(C30*C25*(C20^(C30-1))*C27^C31-C28*C24*(C20^(C28-1))*C27^C29-1/C5)						
38	Smooth pasting 2B	EQ 34	0.000	(C31*C25*C20^C30*(C27^(C31-1))-C29*C24*C20^C28*(C27^(C29-1))+1/C6)						
39	Solution quadrant 1	EQ 26	0.000							
40	Solution quadrant 2	EQ 28	0.000							
41		Sum	0.000	**Solver**						
42		SOLVER: SET C41=0, CHANGING C24:C31								
43		SPREAD	97.04							
44	EQ 26	0.5*C7^2*C28*(C28-1)+0.5*C8^2*C29*(C29-1)+C9*C7*C8*C28*C29+C28*(C10-C5)+C29*(C10-C6)-C10								
45	EQ 28	0.5*C7^2*C30*(C30-1)+0.5*C8^2*C31*(C31-1)+C9*C7*C8*C30*C31+C30*(C10-C5)+C31*(C10-C6)-C10								
46	V_1(x,y)	(C24*C3^C28*C4^C29+C3/C5-C4/C6)								
47	V_2(x,y)	(C25*C3^C30*C4^C31)								

Numerical Illustrations

Figure 4 is an Excel spreadsheet of the simultaneous solution of the ten equations, assuming x_{12}=x_{21}, and deriving the trigger for cost y_{12}>x_{12} that would justify suspension, and the trigger for cost y_{21}<x_{21} that would justify re-starting operations. Note the spreads are very similar to the spreads for the best of two outputs, when the operating cost is constant, due to the similar power function parameter values, but the asset value in operating mode is much lower.

Figure 5 shows the sensitivity of the spreads to changes in the output price volatility, which is similar to Figure 2.

Figure 5

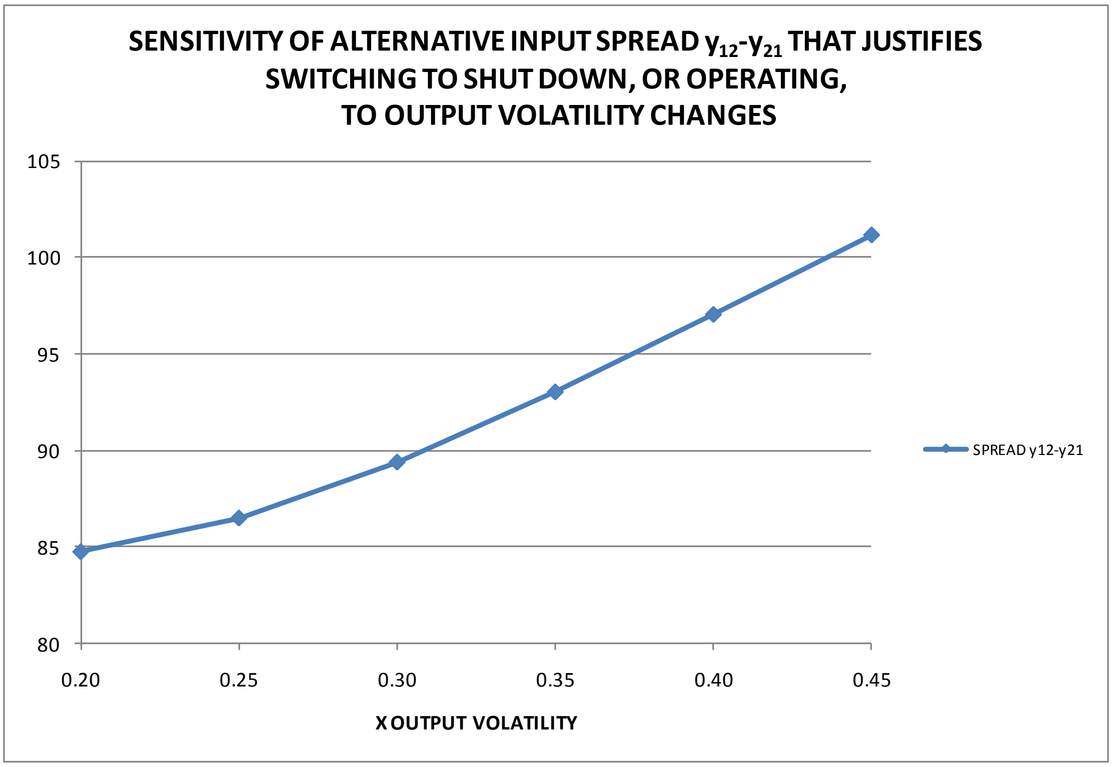


Figure 6

	A	B	C	D	E	F	G	H
1	**SINGLE American Perpetual IN-OUT SWITCH Option Non-homogeneous**							
2		INPUT			IN_OUT_JAN2011			
3	PRICE	x	100					
4	COST	y	50					
5	Convenience yield of x	δ_x	0.04					
6	Convenience yield of y	δ_y	0.04					
7	Volatility of x	σ_x	0.40					
8	Volatility of y	σ_y	0.30					
9	Correlation x with y	ρ	0.50					
10	Risk-free interest rate	r	0.05					
11	Switching cost from x to y	S_{12}	50					
12	Switching boundary OP to SHUT	x_{12}	100					
13		SOLUTION		OPTION	OPERATING			
14	Asset value in operating mode	**V_1(x,y)**	**1630.84**	**380.84**	**1250.00**			
15		A	9.89					
16	Switching boundary OP to SHUT	y_{12} (x)	**337.04**					
17	Solution quadrant	β_{11}	-0.42545	*must be negative*				
18	Solution quadrant	β_{12}	1.43396	*must be positive*				
19		EQUATIONS						
20	Value matching 1	EQ 35	0.000					
21	Smooth pasting 1A	EQ 36	0.000					
22	Smooth pasting 1B	EQ 37	0.000					
23	Solution quadrant 1	EQ 38	0.000					
24		Sum	0.000	**Solver**				
25		SOLVER: SET C24=0, CHANGING C15:C18						
26	SPREAD		287.04					
27	V_1(x,y)	(C15*C3^C17*C4^C18+C3/C5-C4/C6)						
28	EQ 35	(C15*C12^C17*C16^C18+C12/C5-C16/C6+C11)						
29	EQ 36	(C17*C15*(C12^(C17-1))*C16^C18+1/C5)						
30	EQ 37	(C18*C15*C12^C17*(C16^(C18-1))-1/C6)						
31	EQ 38	0.5*C7^2*C17*(C17-1)+0.5*C8^2*C18*(C18-1)+C9*C7*C8*C17*C18+C17*(C10-C5)+C18*(C10-C6)-C10						

Figure 6 shows that the single switching boundary is 191% greater than for multiple switching, with similar parameter values. The asset value V_{1S} is 5% lower compared to multiple switching, because the option to shut without any re-starting is almost 20% lower than the multiple switching option.

4. MULTIPLE INPUT SWITCHING WITH CONSTANT SWITCHING COSTS

The spot price X_I for feedstock $I \in \{1,2\}$ is assumed to follow a geometric Brownian motion process with drift:

$$dX_I = \alpha_I X_I dt + \sigma_I X_I dz_I \qquad (1)$$

where α_I is its instantaneous drift rate, σ_I is the known instantaneous volatility rate, and dz_I is the increment of a standard Wiener process. Dependence between the two spot price variables is described by the instantaneous covariance term $\rho\sigma_1\sigma_2$ where $\mathrm{Cov}\left[dX_1, dX_2\right] = \rho\sigma_1\sigma_2 X_1 X_2 dt$ and $|\rho| \le 1$.

By assuming complete markets, standard contingent claims analysis can be applied to the plant value to determine its risk neutral valuation relationship, expressed for feedstock $I \in \{1,2\}$ by the partial differential equation:

$$\begin{aligned} &\tfrac{1}{2}\sigma_1^2 X_1^2 \frac{\partial^2 F_I}{\partial X_1^2} + \tfrac{1}{2}\sigma_2^2 X_2^2 \frac{\partial^2 F_I}{\partial X_2^2} + \rho\sigma_1\sigma_2 X_1 X_2 \frac{\partial^2 F_I}{\partial X_1 \partial X_2} \\ &\qquad + \theta_1 X_1 \frac{\partial F_I}{\partial X_1} + \theta_2 X_2 \frac{\partial F_I}{\partial X_2} - rF_I + \left(Y_0 - X_I\right) = 0. \end{aligned} \tag{2}$$

where r is the risk-free rate of interest, θ_I denotes the risk-adjusted drift rates for feedstock $I \in \{1,2\}$, and Y_0 is the constant output price. For convenience and simplicity, assume r–θ=δ.

For multiple reciprocal (back and forth) switching, the solution for (2) is:

$$F_1 = A_{14} X_1^{\beta_{14}} X_2^{\eta_{14}} + \frac{Y_0}{r} - \frac{X_1}{r - \theta_1}, \tag{3}$$

$$F_2 = A_{22} X_1^{\beta_{22}} X_2^{\eta_{22}} + \frac{Y_0}{r} - \frac{X_2}{r - \theta_2}. \tag{4}$$

for $I \in \{1,2\}$, where A_I, β_I and η_I are generic parameters whose values have to be determined, using the following characteristic equation for $I \in \{1,2\}$:

$$Q_I\left(\beta_I, \eta_I\right) = \tfrac{1}{2}\sigma_1^2\beta_I\left(\beta_I - 1\right) + \tfrac{1}{2}\sigma_2^2\eta_I\left(\eta_I - 1\right) + \rho\sigma_1\sigma_2\beta_I\eta_I + \theta_1\beta_I + \theta_2\eta_I - r = 0. \tag{5}$$

The remaining unknown parameters that are contained in (3) and (4) are determined by the conditions that have to be fulfilled at the instantaneous switching event. The value matching condition requires that at the optimal switching event the total plant value for the incumbent feedstock is equal to the value of switching, which is the difference between the total plant value for the substitute feedstock and the fixed investment cost required for the switch.

The two value matching relationships are expressed respectively as:

$$F_1\left(\hat{X}_{12},\hat{X}_{22}\right)=F_2\left(\hat{X}_{12},\hat{X}_{22}\right)-K_{12}, \tag{6}$$

$$F_2\left(\hat{X}_{11},\hat{X}_{21}\right)=F_1\left(\hat{X}_{11},\hat{X}_{21}\right)-K_{21}, \tag{7}$$

for $\hat{X}_{12}>\hat{X}_{22}$ and $\hat{X}_{11}<\hat{X}_{21}$. From (3) and (4), the value matching relationships (6) and (7) become respectively:

$$A_{14}\hat{X}_{12}^{\beta_{14}}\hat{X}_{22}^{\eta_{14}}-\frac{\hat{X}_{12}}{r-\theta_1}=A_{22}\hat{X}_{12}^{\beta_{22}}\hat{X}_{22}^{\eta_{22}}-\frac{\hat{X}_{22}}{r-\theta_2}-K_{12}, \tag{8}$$

$$A_{22}\hat{X}_{11}^{\beta_{22}}\hat{X}_{21}^{\eta_{22}}-\frac{\hat{X}_{21}}{r-\theta_2}=A_{14}\hat{X}_{11}^{\beta_{14}}\hat{X}_{21}^{\eta_{14}}-\frac{\hat{X}_{11}}{r-\theta_1}-K_{21}. \tag{9}$$

The terms $\frac{X_1}{r-\theta_1}$ and $\frac{X_2}{r-\theta_2}$ specify the value of the cost of operating in perpetuity with feedstocks 1 and 2, respectively, when X_1 and X_2 represent the prevailing prices. The terms $A_{14}X_1^{\beta_{14}}X_2^{\eta_{14}}$ and $A_{22}X_1^{\beta_{22}}X_2^{\eta_{22}}$ denote, respectively, the value of the option to switch from the incumbent feedstock 1 to the substitute 2 and from the incumbent feedstock 2 to 1, when the incumbent is feedstock 1 (the subscripts indicate the quadrant of an elliptical function Q of the two parameters, see Adkins and Paxson, 2011).

Associated with the two value matching relationships, (8) and (9), there are four smooth pasting conditions:

$$\beta_{14}A_{14}\hat{X}_{12}^{\beta_{14}-1}\hat{X}_{22}^{\eta_{14}}-\frac{1}{r-\theta_1}=\beta_{22}A_{22}\hat{X}_{12}^{\beta_{22}-1}\hat{X}_{22}^{\eta_{22}}, \tag{10}$$

$$\eta_{14}A_{14}\hat{X}_{12}^{\beta_{14}}\hat{X}_{22}^{\eta_{14}-1}=\eta_{22}A_{22}\hat{X}_{12}^{\beta_{22}}\hat{X}_{22}^{\eta_{22}-1}-\frac{1}{r-\theta_2}, \tag{11}$$

$$\beta_{22}A_{22}\hat{X}_{11}^{\beta_{22}-1}\hat{X}_{21}^{\eta_{22}}=\beta_{14}A_{14}\hat{X}_{11}^{\beta_{14}-1}\hat{X}_{21}^{\eta_{14}}-\frac{1}{r-\theta_1}, \tag{12}$$

$$\eta_{22}A_{22}\hat{X}_{11}^{\beta_{22}}\hat{X}_{21}^{\eta_{22}-1}-\frac{1}{r-\theta_2}=\eta_{14}A_{14}\hat{X}_{11}^{\beta_{14}}\hat{X}_{21}^{\eta_{14}-1}. \tag{13}$$

Substituting the various individual switching option values in the value matching relationships (8) and (9) yields, respectively:

$$\frac{\hat{X}_{12}}{r-\theta_1}\left[1-\frac{(\eta_{22}-\eta_{14})}{\Delta}\right]-\frac{\hat{X}_{22}}{r-\theta_2}\left[1-\frac{(\beta_{14}-\beta_{22})}{\Delta}\right]=K_{12}, \tag{14}$$

$$\frac{\hat{X}_{21}}{r-\theta_2}\left[1-\frac{(\beta_{14}-\beta_{22})}{\Delta}\right]-\frac{\hat{X}_{11}}{r-\theta_1}\left[1-\frac{(\eta_{22}-\eta_{14})}{\Delta}\right]=K_{21}\,. \tag{15}$$

where $\Delta=\beta_{14}\eta_{22}-\beta_{22}\eta_{14}$. Justified switching occurs when the difference between the values of operating the incumbent and substitute feedstocks, measured as perpetuities of their respective prices, times the wedge factors, is sufficient to compensate for the switching cost. Specifying the wedge factors by Ω_{IJ} in a similar way as for the X_{IJ}, then:

$$\Omega_{12}=1-\frac{(\eta_{22}-\eta_{14})}{\Delta} \text{ and } \Omega_{22}=1-\frac{(\beta_{14}-\beta_{22})}{\Delta} \tag{16}$$

$$\Omega_{21}=1-\frac{(\beta_{14}-\beta_{22})}{\Delta} \text{ and } \Omega_{11}=1-\frac{(\eta_{22}-\eta_{14})}{\Delta}\,.$$

The wedge factors that are used to adjust the respective values are positive and less than one. They indicate the difference between the real-option and Jevons-Marshallian solutions. This real-option solution contrasts with the equivalent Marshallian rule, which asserts that a switch between the feedstocks is economically justified whenever the incremental value is just sufficient to compensate for the switching cost.

The two value matching relationships, (8) and (9), the four associated smooth pasting conditions, (10) – (13), and the two characteristic equations, (5A and 5B), collectively constitute the general model for the value of an active productive process that embodies a switching option to exchange the substitute feedstock for the incumbent. These eight equations are sufficient to determine the two discriminatory boundaries separating the continuance from the switching region for each of the two possible incumbent feedstocks. It is convenient to observe the current values for $\hat{X}_{21}$, $\hat{X}_{22}$ and then solve the eight model equations simultaneously for $\hat{X}_{11}$, $\hat{X}_{12}$, β_{14}, η_{14}, β_{22}, η_{22} (and by implication A_{14} and A_{22}).

The Single Switch Model

When there is only a single switch opportunity, the option to switch back to the incumbent is not available, so its value is eliminated from the value-matching relationship. Note that in this model, the incumbent could be either input. Consequently, the two value-matching relationships, and (8) and (9) become respectively:

$$A_{14s}\hat{X}_{12s}^{\beta_{14s}}\hat{X}_{22s}^{\eta_{14s}}-\frac{\hat{X}_{12s}}{r-\theta_1}=-\frac{\hat{X}_{22s}}{r-\theta_2}-K_{12}, \tag{17}$$

$$A_{22s}\hat{X}_{11s}^{\beta_{22s}}\hat{X}_{21s}^{\eta_{22s}}-\frac{\hat{X}_{21s}}{r-\theta_2}=-\frac{\hat{X}_{11s}}{r-\theta_1}-K_{21}, \tag{18}$$

where the subscript s indicates the single opportunity representation, and $A_{14s},A_{22s}\geq 0$, $\beta_{14s},\eta_{22s}>0$, $\beta_{22s},\eta_{14s}<0$. The determination of the first timing boundary is explained in full below, while only the result for the second is given.

The two smooth-pasting conditions associated with (17) can be expressed as:

$$A_{14s}\hat{X}_{12s}^{\beta_{14s}}\hat{X}_{22s}^{\eta_{14s}}=\frac{\hat{X}_{12s}}{\beta_{14s}\left(r-\theta_1\right)}=-\frac{\hat{X}_{22s}}{\eta_{14s}\left(r-\theta_2\right)}>0. \tag{19}$$

Eliminating $A_{14s}\hat{X}_{12s}^{\beta_{14s}}\hat{X}_{22s}^{\eta_{14s}}$, (17) becomes:

$$\frac{\hat{X}_{12s}}{r-\theta_1}-\frac{\hat{X}_{22s}}{r-\theta_2}=K_{12}+\frac{\hat{X}_{12s}}{\beta_{14s}\left(r-\theta_1\right)}. \tag{20}$$

For a single opportunity switch between feedstocks to be economically justified, the incremental value rendered by the switch has to compensate not only for the switching cost but also for the value of the option to switch once.

There are three equations constituting the single opportunity switch model for feedstock 1 as the incumbent. These are (i) the reduced form value-matching relationship (20), (ii) the smooth-pasting condition (19) and (iii) the characteristic root equation (5) $Q_1\left(\beta_{14s},\eta_{14s}\right)=0$. The second timing boundary for feedstock 2 as the incumbent is determined in a similar way. The reduced form value-matching relationship is:

$$\frac{\hat{X}_{21s}}{r-\theta_2}-\frac{\hat{X}_{11s}}{r-\theta_1}=K_{21}+\frac{\hat{X}_{21s}}{\eta_{22s}\left(r-\theta_2\right)}, \tag{21}$$

and the smooth-pasting conditions can be expressed as:

$$\frac{\hat{X}_{21s}}{\eta_{22s}\left(r-\theta_2\right)}=-\frac{\hat{X}_{11s}}{\beta_{22s}\left(r-\theta_1\right)}. \tag{22}$$

The timing boundary relating $\hat{X}_{21s}$ and $\hat{X}_{11s}$ is derived from (21), (22) and (6) $Q_2\left(\beta_{22s},\eta_{22s}\right)=0$. From (19) and (20) and (21) and (22), rearranged,

$$\frac{\hat{X}_{12}}{r-\theta_1}\left[1-\frac{1}{\beta_{14}+\eta_{14}}\right]-\frac{\hat{X}_{22}}{r-\theta_2}\left[1-\frac{1}{\beta_{14}+\eta_{14}}\right]=K_{12}, \tag{23}$$

$$\frac{\hat{X}_{21}}{r-\theta_2}\left[1-\frac{1}{\eta_{22}+\beta_{22}}\right]-\frac{\hat{X}_{11}}{r-\theta_1}\left[1-\frac{1}{\eta_{22}+\beta_{22}}\right]=K_{21}, \tag{24}$$

Numerical Illustrations

For the multiple input switching model, in Figure 7, based on a positive correlation of .5, when the feedstock-in-use 2 is 50, it pays to switch to feedstock 1 if its price is 24.6 or below. When the feedstock-in-use 1 is 92.3, it pays to switch to feedstock 2 if its price is 50 or below. The first wedge factor Ω_{12}=.071, the second wedge factor Ω_{22}=.091, so (X_{12}(92.3)/.04)*Ω_{12}-(X_{22}(50)/.04)*Ω_{22}=50, using (14). Using (15), the first wedge factor Ω_{11}=.091, the second wedge factor Ω_{21}=.071, so (X_{21}(50)/.04)*Ω_{11}-(X_{11}(24.6)/.04)*Ω_{21}=70.

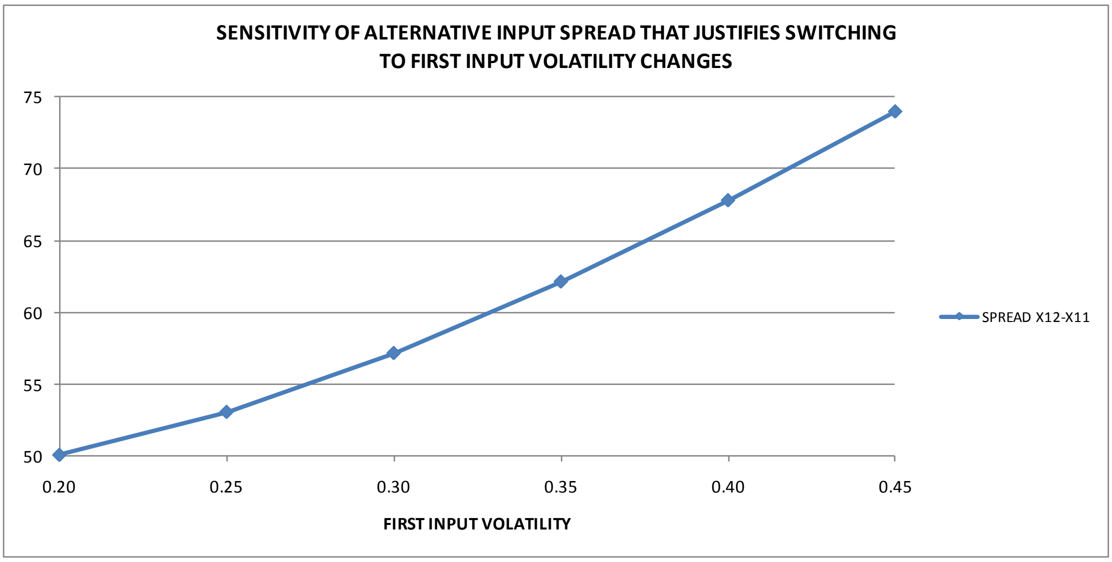
Applying (3) and (4), if Y_0 =100 and X_1=X_2=50, F_1=626+750=1376, F_2=613+750=1363. The first part of the LHS is the value of the option to switch, and the second part the production value. Note that the switching options (value of feedstock flexibility) are significant. Since F_1>F_2, start with X_1, since the switching option value for F_1 is greater than for F_2.

Figure 8 illustrates the sensitivity of the switching boundaries of the quasi-analytical solution for continuous switching to changes in the first incumbent input volatility. Switching boundaries are further apart when volatilities are higher. This is consistent with general real option theory because uncertainty is taken into account which delays switching in order to gain more information.

Figure 7

	A	B	C	D	E	F	G	H	I	J
1		**INPUT**								
2		**Multiple Switch Model**								
3		Input								
4	K12	50								
5	K21	70								
6	σ1	40.0%								
7	σ2	30.0%								
8	ρ	0.5								
9	r	5%								
10	θ1	1%								
11	θ2	1%								
12	r−θ1	4%								
13	r−θ2	4%								
14										
15	A14	9.58								
16	A22	8.04								
17	X21	50.00								
18	X22	50.00								
19		Solution								
20	X11	**24.5467**								
21	X12	**92.3482**								
22	β14	1.4215								
23	η14	-0.3528								
24	β22	-0.3648								
25	η22	1.4727								
26	A14	9.5798								
27	A22	8.0361								
28	Equations									
29	EQ 8	0.0000								
30	EQ 9	0.0000								
31	EQ 10	0.0000								
32	EQ 11	0.0000								
33	EQ 12	0.0000								
34	EQ 13	0.0000								
35	EQ 5A	0.0000								
36	EQ 5B	0.0000								
37	SUM	0.0000								
38	SOLVER B37=0, Changing B20:B27									
39	EQ 8	B26*(B21^B22)*(B18^B23)-B21/B12-(B27*(B21^B24)*(B18^B25)-B18/B13-B4)								
40	EQ 9	B27*(B20^B24)*(B17^B25)-B17/B13-(B26*(B20^B22)*(B17^B23)-B20/B12-B5)								
41	EQ 10	B22*B26*(B21^(B22-1))*(B18^B23)-1/B12-(B24*B27*(B21^(B24-1))*(B18^B25))								
42	EQ 11	B23*B26*(B21^B22)*(B18^(B23-1))-(B25*B27*(B21^B24)*(B18^(B25-1))-1/B13)								
43	EQ 12	B24*B27*(B20^(B24-1))*(B17^B25)-(B22*B26*(B20^(B22-1))*(B17^B23)-1/B12)								
44	EQ 13	B25*B27*(B20^B24)*(B17^(B25-1))-1/B13-(B23*B26*(B20^B22)*(B17^(B23-1)))								
45	EQ 5A	0.5*(B6^2)*B22*(B22-1)+0.5*(B7^2)*B23*(B23-1)+B8*B6*B7*B22*B23+B22*B10+B23*B11-B9								
46	EQ 5B	0.5*(B6^2)*B24*(B24-1)+0.5*(B7^2)*B25*(B25-1)+B8*B6*B7*B24*B25+B24*B10+B25*B11-B9								
47	Y0	100								
48	Suppose X21=X22=50 at start,		Option	Production						
49	F1=EQ 3	**1376.46**	**626.46**	**750.00**						
50	F2=EQ 4	**1362.64**	**612.64**	**750.00**						
51	Since F1>F2, start with X1, more volatile, but higher conyield, so lower perpetual production cost.									
52	Switching option is greater for X1.									
53		Ω1	Ω2							
54	WEDGE 1-2 EQ 14	0.071	0.091							
55	WEDGE 2-1 EQ 15	0.091	0.071							
56										
57	F1=EQ 3	B47/B9+B26*(B17^B22)*(B18^B23)-B17/B12								
58	F2=EQ 4	B47/B9+B27*(B17^B24)*(B18^B25)-B18/B13								
59	SPREAD	**67.80**								

Figure 8



When switching is only possible from one input to the other but not back, the switching trigger is very much higher as shown in Figure 9 because the decision cannot be reversed. The asset value before the one-time switch is EQ 3 for F1 if starting with X1, but if starting with X2 EQ 4 for F2. After the switch, the asset value in either case is just the production value. The asset values are lower than in the multiple case, with the highest single switching value some 8% lower compared to the highest multiple switching value.

Figure 9

	A	B	C	D	E	F	G	H
1	**INPUT: CASE**							
2	**Single Switch Model**							
3		Input						
4	K12	50						
5	K21	70						
6	$\sigma 1$	40.0%						
7	$\sigma 2$	30.0%						
8	ρ	0.5						
9	r	5%						
10	$\theta 1$	1%						
11	$\theta 2$	1%						
12	$r-\theta 1$	4%						
13	$r-\theta 2$	4%						
14								
15	A14	9.63						
16	A22	7.42						
17								
18	X21	50.0000						
19	X22	50.0000						
20		Solution						
21	X11	**13.0331**						
22	X12	**173.3528**						
23	$\beta 22$	-0.3815						
24	$\eta 22$	1.4634						
25	$\beta 14$	1.4285						
26	$\eta 14$	-0.4120						
27								
28	Equations							
29	EQ 20	0.0000						
30	EQ 19	0.0000						
31	EQ 21	0.0000						
32	EQ 22	0.0000						
33	EQ 6A	0.0000						
34	EQ 6B	0.0000						
35	SUM	0.0000						
36	Solver B35=0, Changing B21:B26							
37	EQ 20	((B25-1)/B25)*(B22/B12)-(B19/B13)-B4						
38	EQ 19	B22/B12-((B26-1)/B26)*(B19/B13)-B4						
39	EQ 21	(B18/B13)*((B24-1)/B24)-(B21/B12)-B5						
40	EQ 22	B18/B13-(B21/B12)*((B23-1)/B23)-B5						
41	EQ 6A	0.5*(B6^2)*B25*(B25-1)+0.5*(B7^2)*B26*(B26-1)+B8*B6*B7*B25*B26+B25*B10+B26*B11-B9						
42	EQ 6B	0.5*(B6^2)*B23*(B23-1)+0.5*(B7^2)*B24*(B24-1)+B8*B6*B7*B23*B24+B23*B10+B24*B11-B9						
43	SUM	SUM(ABS(B29:B34))						
44								
45	A14	(B22^(1-B25))/((B19^B26)*B25*B12)						
46	A22	-(B21^(1-B23))/((B18^B24)*B23*B12)						
47	Y0	100						
48	Suppose X1=X2=50 at start		Option	Production				
49	F_{1S}=EQ 3	**1263.63**	**513.63**	**750.00**				
50	F_{2S}=EQ 4	**1261.45**	**511.45**	**750.00**				
51	Since F2<F1, start with X1, more volatile, same conyield, so same perpetual production cost.							
52	Switching option is greater for X1, enough to offset same production cost.							
53		$\Omega 1$	$\Omega 2$					
54	WEDGE 1-2 EQ23	0.016	0.016					
55	WEDGE 2-1 EQ24	0.076	0.076					
56	PROOF WEDGE1-2	0.0000						
57	PROOF WEDGE2-1	0.0000						

5 Policy and Strategy Implications

There are a number of stakeholders shown in Figure 10 whose best decisions should be based on these switching models.

Figure 10

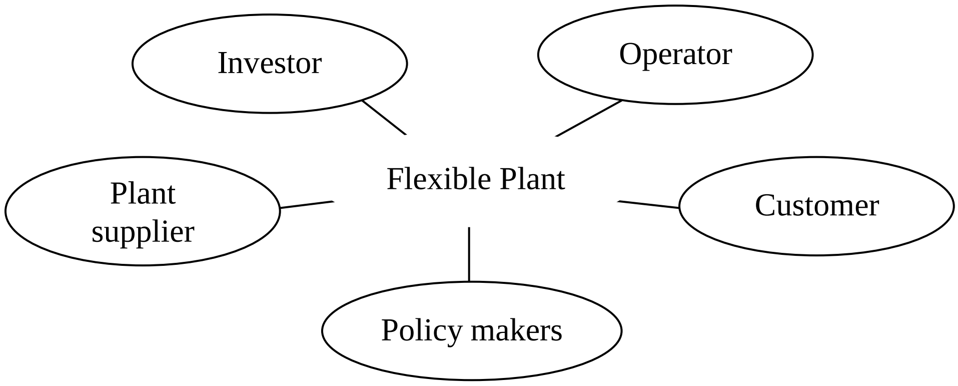


Figure 11

	TWO OUTPUTS	IN-OUT	TWO INPUTS
MULTIPLE			
PRODUCTION	1500.00	1250.00	750.00
V1	2780.37	1718.27	1376.46
V2	2781.76	1649.16	1362.64
SPREAD	97.04	97.04	67.80
SINGLE			
PRODUCTION	1500.00	1250.00	750.00
V1	2528.98	1630.84	1263.63
V2	1500.00	0.00	1261.45
SPREAD	237.04	287.04	160.32

Investors

As shown in Figure 11, the real option value of all of these flexible facilities is substantially greater than the present value of current production (= inflexible facilities), at the current assumed input and output price levels. Note the focus of alert investors is on choosing the appropriate model and on forecasting input and output price volatilities and correlations. A myopic investment analyst using net present values will probably undervalue flexible plants. Analysts may not have access to plant operating or switching costs, or indeed knowledge of any flexibility inherent in existing facilities, due conceivably to inadequate accounting disclosures, not currently required by accounting standard setting committees. Of course, realistic analysts may doubt that the chief option managers of flexible facilities will be aware of the potential optionality, or indeed make switches at appropriate times, so the Marshallian values might reflect a realistic allowance for management shortfalls.

Chief Real Options Manager

The alert chief options manager ("CROM") is aware of input and output switching opportunities, the amount of switching costs, and periodically observes input and output prices, convenience yields (or proxies), updates expected volatilities and correlations, and so updates Figure 11 appropriately. Observed current spreads between input/output prices are compared to the updated triggers for switching, perhaps based on simple approximate linear rules over short or stable periods. Naturally part of the appropriate compensation for the CROM should be based on awareness of these opportunities, and performance in making actual input and output switches at appropriate times.

Originally, the CROM would have calculated the value of a flexible plant V_1 or V_2, compared to an inflexible facility, which also indicates the warranted extra investment cost for facility flexibility. It would not be difficult to consider trade-offs for any deterministic lower efficiency due to the flexibility capacity.

Plant Suppliers

Originally, suppliers of facilities to the CROM would have calculated the value of a flexible plant V_1 or V_2, compared to an inflexible facility, which also indicates the warranted extra investment price that could be charged for facility flexibility. With the illustrated parameter values, a hypothetical multiple switch facility is worth only some 5-10% more than a single switch plant, but much more than an inflexible facility. In designing flexible facilities, it would not be difficult to consider trade-offs for any lower efficiency due to the flexibility capacity against increased building costs.

Customers

Output customers may be aware of the limitations, or capacities, of producers to switch to higher price products, opportunistically, or to alternative lower price inputs when appropriate. Input suppliers may become cautious with buyers, who switch sources optimally. Other customers might seek long-term agreements mitigating the shifts in output and input prices implied in using real option approaches for operating flexible facilities.

Policy Makers

Taxpayers beware. There will be national producers without flexible facilities, or not aware of needing to change output prices, and input sources, as the economic environment changes. Those producers priced out of the market will seek government barriers for other producers, or input/output subsidies as conditions change.

6 Applications

Flexibility between outputs and inputs is particularly relevant in volatile commodity markets, or where free trade allows new entrants, cheaper inputs, or more valuable outputs. Think of the many applications for substitute outputs, substitute inputs, or alternative inputs and outputs. Dockendorf and Paxson (2011) examine further processed chemical products as essentially output alternatives. They note alternative uses of other types of facilities, such as multiuse sports or entertainment or educational facilities, transportation vehicles for passengers or cargo, rotating agricultural crops, and solar energy used for electricity or water desalination. Adkins and Paxson (2011) note there are numerous energy switching opportunities, such as palm or rape oil in biodiesel production, gas-oil-hydro-coal in electricity generation, that are reciprocal energy input switching options. There are several examples of stochastic output and input prices, such as the "crack" spread for gasoline-heating oil as outputs for crude oil refineries, the "crush" spread for soya meal and soya oil as outputs for soya bean refineries, and ethanol the output for corn processing facilities.

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EXERCISE 12.1

George Gamble owns and operates the HBS renewable biodiesel facility, which can be switched once (at a cost of \$ .50) between canola and palm oil inputs. He is currently using canola at \$1 per unit, as is palm oil, both have volatility of 20%, .5 correlated, convenience yield of 1%, r=5%, and output sells for \$3 per unit. If palm oil is still $\$1=X_{22}$, at what increased canola price should he switch to palm oil (X_{12})? George figures that $\beta=2.0032$, $\eta=-.9835$. HINT: see input wedge EQ 23, page 20.

EXERCISE 12.2

Eventually Marianne switches to the aged, kind Colonel Brandon (annual income £ 100=y) over the exciting, handsome Willoughby (ignoring his real marriage options and gambling debts, income £$70=x_{12}$) even though the switching costs (£100 in emotional pain) are great. Both are volatile (20%), are 100% negatively correlated, are not very convenient (4%), and their fortunes are invested in gilts which yield 5%. The Colonel calculates that $\beta_{11}=-.357$, $\beta_{12}=1.378$, $y_{12}/x_{12}=-\beta_{12}/\beta_{11}$, and $A=(-1/\delta_x)/(\beta_{11}*x_{12}^{(\beta 11-1)} * y_{12}^{\beta 12})$. Is she right even if she receives her husband's income? HINT: see EQ 5, page 5.

EXERCISE 12.3

Julia Smith, who is nearly immortal, is taking the Real Options course (RO), and believes it offers the same career benefit (1 per annum) as Investment Analysis (IA) (1). Both are very easy with no cost or effort, but both are uncertain ($\sigma_{RO}=\sigma_{IA}=10\%$), $\delta_{RO}=\delta_{IA}=5\%$, $\rho=0$, r=5%. Professors Great and Smart have agreed that Julia may switch courses once anytime for a small fee of .2. They report that $\beta_{11}=-1.78$ and $\beta_{12}=2.80$. What IA benefits would justify switching ? The switching boundary equals $-\beta_{12}/\beta_{11}$., and $A=(-1/\delta_x)/(\beta_{11}*y_{12}$^$\beta_{12})$. What is the current value of taking RO? HINT: see EQ 5, page 5.

PROBLEM 12.4

Lucy Steele is secretly engaged to Edward Ferrar, who has a yearly income of ₤10,000. When Edward's mother learns of this engagement, she transfers his inheritance to the younger brother Robert, who is not handsome, so the cost to Lucy of switching to Robert is significant (₤ XXX). Supposing that Edward will eventually obtain ₤7000 annually from Colonel Brandon, would a switching cost of ₤ 3000 justify Lucy's switch, if this can be forever delayed? Both Edward and Robert are volatile (20%), not correlated, are not very convenient (4%), and their fortunes are invested in gilts which yield 5%.

PROBLEM 12.5

George Gamble owns and operates the HBS renewable biodiesel facility, which can switch back and forth between canola and palm oil inputs. He is currently using canola at $1 per unit, as is palm oil, both have volatility of 20%, .5 correlated, convenience yield of 1%, r=5%, switching cost $ .50 and output sells for $3 per unit. When should he switch to palm oil, and then back to canola? What is HBS worth?

PROBLEM 12.6

Julia Smith, who is nearly immortal, believes the Real Option course (RO) offers a career benefit of 10 per annum compared to Investment Analysis (IA) (8). Both are hard courses with a cost of 1. Both are uncertain ($\sigma_{RO}=\sigma_{IA}=10\%$), $\delta_{RO}=\delta_{IA}=5\%$, $\rho=0$, r=5%. Professors Great and Smart have agreed that Julia may switch courses back and forth anytime for a fee of 1. Julia believes the perceived value of these courses to her will fluctuate throughout her job interviews. What IA benefits would justify switching, or once switched what RO benefits would justify switching back. What is the current value of taking RO?